

# CONVERGENCE IN MEASURE UNDER FINITE ADDITIVITY

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**ABSTRACT.** We investigate the possibility of replacing the topology of convergence in probability with convergence in  $L^1$ . A characterization of continuous linear functionals on the space of measurable functions is also obtained.

## 1. INTRODUCTION AND NOTATION

This paper investigates some properties of the space  $L^0(\lambda)$  of  $\lambda$ -measurable, real valued functions on some set  $\Omega$ , where  $\lambda$  is a bounded, finitely additive set function defined on an algebra  $\mathcal{A}$  of subsets of  $\Omega$ , i.e.  $\lambda \in ba(\mathcal{A})$ . We first characterize in section 2 the dual space of  $L^0(\lambda)$  and study some of its properties, particularly positivity. In the following section 3 we investigate several boundedness conditions and their implications for continuity. Eventually, in section 4 we study convergence properties of sequences. Section 2 is quite independent from the following two.

Although being an entirely standard and widely used concept in probability and mathematical statistics, convergence in measure is much less popular in analysis, even assuming countable additivity. It is known that the corresponding topology is completely metrizable but, in general, not separable; moreover, the topology is not linear so that some useful tools such as separation theorems are not available. Actually, even a characterization of continuous linear functionals is missing. Finite additivity introduces additional complications inducing, e.g., the corresponding topology to be non complete.

The main idea of this paper is to show that some of the techniques developed in the classical setting are still available under finite additivity, particularly measure changes. In particular we show that, upon replacing the original measure with another suitably chosen but *near* to it, the topology of convergence in measure may be replaced by the  $L^1$  topology. We obtain thus in Corollaries 3 and 4 conditions under which continuous,  $L^0(\lambda)$  valued operators are continuous as maps on  $L^1(\mu)$  upon a convenient change of measure. Likewise, Theorem 4 proves, that a sequence converging in measure admits, after such a change, a subsequence converging in  $L^1$ . We also obtain in Theorem 5 a partial analogous under finite additivity of the celebrated lemma of Komlós.

In the notation as well as the terminology on finitely additive measures and integrals we mainly follow Dunford and Schwarz [6] although we prefer the symbol  $|\mu|$  of [2] to denote the total variation

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measure associated with  $\mu \in ba(\mathcal{A})$ . The integral of  $f \in L^1(\mu)$  will be denoted at will as  $\int f d\mu$  or  $\mu(f)$  but always as  $\mu_f$  when considered itself as a set function.

Some special subfamilies of  $ba(\mathcal{A})$  will be treated. In particular, we denote by (i)  $ba_0(\mathcal{A}, \lambda)$ , (ii)  $ba_1(\mathcal{A}, \lambda)$ , (iii)  $ba_\infty(\mathcal{A}, \lambda)$  and (iv)  $ba_*(\mathcal{A}, \lambda)$  the classes of those set functions  $\mu \in ba(\mathcal{A})$  such that (i)  $\mu$  has finite range, (ii)  $d\mu/d\lambda \in L^1(\lambda)$ , (iii)  $|\mu| \leq c|\lambda|$  for some  $c \in \mathbb{R}_+$  and (iv)  $\mu \in ba_\infty(\mathcal{A}, \lambda)$  and  $|\mu|(A) = 0$  if and only if  $|\lambda|(A) = 0$ , respectively. In the above defined families the symbol  $ba$  will be replaced by  $\mathbb{P}$  to indicate the intersection of the corresponding family with the set  $\mathbb{P}(\mathcal{A})$  of finitely additive probability measures on  $\mathcal{A}$ .

The linear space of  $\mathcal{A}$ -simple functions, generated by the indicators of sets in  $\mathcal{A}$ , will be indicated by  $\mathcal{S}(\mathcal{A})$  and, when considered as a normed space, will always be endowed with the supremum norm. A sequence  $\langle f_n \rangle_{n \in \mathbb{N}}$  in  $L^0(\lambda)$   $\lambda$ -converges to  $f \in \mathbb{R}^\Omega$  if  $\lim_n |\lambda|^*(|f_n - f| > c) = 0$  for any  $c > 0$ , in which case  $f \in L^0(\lambda)$  too.  $f \in L^0(\lambda)$  if and only if there exists a sequence  $\langle f_n \rangle_{n \in \mathbb{N}}$  in  $\mathcal{S}(\mathcal{A})$  which  $\lambda$ -converges to  $f$ .

The set  $L^0(\lambda)$  of measurable functions is endowed with the metric

$$(1) \quad d(f, g) = \inf \{c + |\lambda|^*(|f - g| > c) : c > 0\}$$

(or equivalently with  $\rho(f, g) = |\lambda|(|f - g| \wedge 1)$ ) and by a bounded subset of  $L^0(\lambda)$  we will always mean a subset bounded with respect to such metric.  $L^0(\lambda)$  is also a vector lattice if we write  $f \geq g$  whenever  $|\lambda|^*(f \leq g - c) = 0$  for all  $c > 0$ . We then say that  $\mathcal{K} \subset L^0(\lambda)$  is lower (resp. upper, order) bounded when  $k \geq f$  (resp.  $k \leq f$ ,  $|k| \leq f$ ) for some  $f \in L^0(\lambda)$  and all  $k \in \mathcal{K}$ .

We will use repeatedly the following, finitely additive version of Tchebiceff inequality where  $f \in L^1(\lambda)_+$ :

$$|\lambda|^*(f > c) = \sup m(f > c) \leq c^{-1} \sup m(f) \leq c^{-1} |\lambda|(f)$$

where the supremum is computed over all  $m \in ba(\sigma\mathcal{A})_+$  which are extensions of  $|\lambda|$ , see [2, 3.3.3].

## 2. LINEAR FUNCTIONALS ON $L^0(\lambda)$

Each  $\lambda \in ba_0(\mathcal{A})$  may be written as a finite sum  $\sum_{n=1}^N \alpha_n \lambda_n$  where, for each  $n$ ,  $\lambda_n \in ba(\mathcal{A})$  takes its values in the set  $\{0, 1\}$ . From [2, proposition 11.1.5] we learn that  $\lambda \in ba_0(\mathcal{A})$  if and only if there exists  $\eta > 0$  such that  $A \in \mathcal{A}$  and  $|\lambda|(A) > 0$  imply  $|\lambda|(A) \geq \eta$ . It is easy to draw from this the conclusion that  $ba_0(\mathcal{A})$  is an ideal, that is that  $m \in ba_0(\mathcal{A})$ ,  $\mu \in ba(\mathcal{A})$  and  $|\mu| \leq |m|$  imply  $\mu \in ba_0(\mathcal{A})$ , and, furthermore, that  $ba_0(\mathcal{A})$  is a convex, extremal subset of  $ba(\mathcal{A})$ .

**Theorem 1.** *There exists an isomorphism between the space of continuous linear functionals on  $L^0(\lambda)$  and the space  $ba_0(\mathcal{A}, \lambda)$  and this is defined implicitly via the identity*

$$(2) \quad \phi(f) = \int f d\mu \quad f \in L^0(\lambda)$$

*Proof.* Continuous linear functionals on  $L^0(\lambda)$  form a vector lattice. Indeed, if  $f \in L^0(\lambda)$  the set  $\mathcal{U}(f) = \{g \in L^0(\lambda) : |g| \leq |f|\}$  is bounded in  $L^0(\lambda)$  and so is any order bounded set  $[h, f] = \{g \in L^0(\lambda) : h \leq g \leq f\}$  given the inclusion  $[h, f] \subset h + \mathcal{U}(f - h)$ . Any continuous linear functional

$\phi$  on  $L^0(\lambda)$  is thus order bounded and the claim follows from [1, Theorem 1.13, p. 12]. There is thus no loss of generality in assuming, as we shall do henceforth, that  $\phi$  is positive. Observe that  $\phi(f) = \lim_n \phi(f \wedge n)$  for each  $f \in L^0(\lambda)_+$  so that the existence of the representation (2) follows from [3, Theorem 1, p. 561].  $\mu \ll \lambda$  is a consequence of  $\phi$  being continuous. Suppose that for each  $n \in \mathbb{N}$  there is  $H_n \in \mathcal{A}$  such that  $0 < \mu(H_n) \leq 2^{-n}$  and let  $G_k \in \mathcal{A}$  be such that  $\mu(G_k^c) + \lambda_\mu^\perp(G_k) < 2^{-k}$ . Then, choosing the integer  $k_n$  large enough and  $H'_n = H_n G_{k_n}$ , we have  $0 < \mu(H'_n) \leq 2^{-n-1}$  and  $\lim_n \lambda(H'_n) = 0$ . If  $f_n = \lambda(H'_n)^{-1} \mathbf{1}_{H'_n}$  then  $\langle f_n \rangle_{n \in \mathbb{N}}$  converges to 0 in  $\lambda$  measure but  $\phi(f_n) = \int f_n d\mu = 1$ , a contradiction. We conclude that for  $n$  large enough  $\mu(A) \leq 2^{-n}$  implies  $\mu(A) = 0$ , i.e.  $\mu \in ba_0(\mathcal{A})$ . Conversely, assume that  $\mu \in ba_0(\mathcal{A}, \lambda)$  and that  $U \subset L^0(\lambda)$  is bounded in  $L^0(\lambda)$ . Then, choosing  $\delta$  accurately,  $|\mu|^*(|f| > \delta) = 0$  so that  $\sup_{f \in U} |\int f d\mu| \leq \delta \|\mu\|$  so that the  $\mu$  integral defines a bounded linear functional on  $L^0(\lambda)$ .  $\square$

Each non trivial, continuous linear functional on  $L^0(\lambda)$  is thus isomorphic to a continuous linear functional on  $L^1(\mu)$ , upon a change of the underlying probability measure. The inclusion  $\mu \in ba_0(\mathcal{A}, \lambda)$  implies also that a set bounded in  $L^0(\lambda)$  is necessarily bounded in  $L^1(\mu)$  or even in  $L^\infty(\mu)$ . Moreover, if  $m$  is countably additive, then so is  $\mu$ . However,  $\lambda$  and  $\mu$  may be very far from one another, e.g. for what concerns null sets. In fact we can prove the following

**Corollary 1.**  *$L^0(\lambda)$  admits a strictly positive linear functional if and only if  $\lambda \in ba_0(\mathcal{A})$ .*

*Proof.* In fact, with the same notation of Theorem 1, in order for  $\phi$  to be strictly positive one should have  $\lambda \sim \mu$  but this implies that, for some  $\delta > 0$ , from  $|\lambda|(F) < \delta$  follows  $|\mu|(F) = 0$  and thus  $|\lambda|(F) = 0$ . On the other hand, if  $\lambda \in ba_0(\mathcal{A})$  then the integral  $\int f d|\lambda|$  is well defined for all  $f \in L^0(\lambda)$  and strictly positive as  $f \in L^0(\lambda)_+$  and  $\int f d|\lambda| = 0$  imply  $\lambda^*(f > c) = 0$  for all  $c > 0$ , i.e.  $f$  is  $\lambda$  null.  $\square$

Thus if  $\lambda \notin ba_0(\mathcal{A})$  there does not exist any strictly positive linear functional, so that if  $\mu$  is as in Theorem 1 one necessarily has sets  $A \in \mathcal{A}$  such that  $\lambda(A) > 0 = \mu(A)$ . The goal of the next section will be to find  $\mu \in ba(\mathcal{A})$  that guarantees the integrability of some subset of  $L^0(\lambda)$  without affecting null sets.

**Corollary 2.** *Let  $\mathcal{A}$  be a  $\sigma$ -algebra,  $\lambda \in ca(\mathcal{A})$  and let  $\phi$  be a continuous linear functional on  $L^0(\lambda)$ . Then either  $\phi = 0$  or  $\lambda$  has atoms.*

*Proof.* Let  $\mu \in ba_0(\mathcal{A}, \lambda)$  be associated with  $\phi$  as in Theorem 1. Under the current assumptions,  $\mu$  admits a Radon Nikodym derivative  $f_\mu \in L^1(\lambda)$ . Let  $\eta > 0$  be such that  $A \in \mathcal{A}$  and  $\mu(A) < \eta$  imply  $\mu(A) = 0$ . Let also  $c < \eta/\|\lambda\|$ . Then  $\mu(f_\mu < c) = \lambda(f \mathbf{1}_{\{f_\mu < c\}}) \leq c\lambda(f_\mu < c) < \eta$  so that  $\mu(f_\mu < c) = 0$  and therefore  $\mu \geq c\lambda$ . But then, if  $\delta$  is such that  $\lambda(A) < \delta$  implies  $\mu(A) < \eta$  we have also the implication  $\lambda(A) = 0$ , so that  $\lambda \in ba_0(\mathcal{A})$ . It follows from countable additivity that  $\lambda$  has atoms.  $\square$

3. BOUNDED SUBSETS OF  $L^0(\lambda)$ 

In this section we obtain some results on  $L^0(\lambda)$  bounded sets, providing conditions under which these sets become bounded in  $L^1$  under a change of the given measure. The technique of changing the underlying probability measure is rather popular in stochastic analysis, e.g. in the study of semimartingale topologies, see [8].

**Lemma 1.** *Let  $\mathcal{K} \subset L^1(\lambda)$  be convex with  $\emptyset \in \mathcal{K}$ . If  $\mathcal{K}$  is bounded in  $L^0(\lambda)$  then there exists  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $\mathcal{K} \subset L^1(\mu)$  and  $\sup_{k \in \mathcal{K}} \mu(k) < \infty$ .*

*Proof.* Denote by  $\overline{\mathcal{C}}$  the closure in  $L^1(\lambda)$  of the set  $\mathcal{C} = \mathcal{K} - \mathcal{S}(\mathcal{A})_+$ . Pick  $A \in \mathcal{A}$  such that  $|\lambda|(A) > 0$  and fix  $x > 0$ . Suppose that  $x\mathbf{1}_A \in \overline{\mathcal{C}}$ . For each  $n \in \mathbb{N}$  there exist then  $k_n \in \mathcal{K}$  and  $h_n \in \mathcal{C}$  such that  $k_n \geq h_n$  and  $|\lambda|(|h_n - 2x\mathbf{1}_A|) < 2^{-n}$ . Thus,

$$\begin{aligned} |\lambda|^*(k_n > x) &\geq |\lambda|^*(h_n > x) \\ &\geq |\lambda|^*(|h_n - 2x| < x) \\ &\geq |\lambda|(A) - |\lambda|^*(A \cap \{|h_n - 2x| \geq x\}) \\ &\geq |\lambda|(A) - |\lambda|^*(|h_n - 2x\mathbf{1}_A| \geq x) \\ &\geq |\lambda|(A) - \delta^{-1}|\lambda|(|h_n - 2x\mathbf{1}_A|) \\ &\geq |\lambda|(A) - x^{-1}2^{-n} \end{aligned}$$

i.e.  $\sup_{k \in \mathcal{K}} |\lambda|^*(|k| > x) \geq |\lambda|(A)$ . Thus, for  $x$  sufficiently high,  $2x\mathbf{1}_A \notin \overline{\mathcal{C}}$ . Then the claim follows from the finitely additive version of Yan Theorem [4, Corollary 8].  $\square$

The most important consequence of the preceding Lemma is the following:

**Theorem 2.** *Let  $\mathcal{K} \subset L^0(\lambda)$  be convex and admit a lower bound. If  $\mathcal{K}$  is bounded in  $L^0(\lambda)$  then there exists  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $\mathcal{K}$  is a bounded subset of  $L^1(\mu)$ . If  $\lambda \in ca(\mathcal{A})$  then the converse is also true.*

*Proof.* Let  $f \in L^0(\lambda)$  be a lower bound for  $\mathcal{K}$  and define the sets

$$\mathcal{K}_0 = \{\alpha(k - f) + \beta|f| : k \in \mathcal{K}, \alpha, \beta \geq 0, \alpha + \beta \leq 1\} \quad \text{and} \quad \mathcal{K}_1 = \{h \wedge k : h \in L^1(\lambda)_+, k \in \mathcal{K}_0\}$$

Observe that  $\mathcal{K}_1$  is a convex subset of  $L^1(\lambda)_+$  with  $\emptyset \in \mathcal{K}_1$ ; moreover,  $\mathcal{K}_1$  is bounded in  $L^0(\lambda)$ . We deduce from Lemma 1 the existence of  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $\mathcal{K}_1 \subset L^1(\mu)$  and  $\sup_{h \in \mathcal{K}_1} \mu(h) < \infty$ . Observe that if  $k \in \mathcal{K}$ , then given the inclusions  $k - f, |f| \in \mathcal{K}_0$  and the inequality  $|k| \wedge n \leq (k - f) \wedge n + |f| \wedge n$  we obtain  $\frac{1}{2}(|k| \wedge n) \in \mathcal{K}_1$  and consequently

$$\mu(|k|) = \lim_n \mu(|k| \wedge n) \leq 2 \sup_{h \in \mathcal{K}_1} \mu(h)$$

If we assume, conversely, the existence of  $\mu$  as in the claim and if  $\lambda$  is countably additive, then  $\mu \gg \lambda$  and, given that  $\mathcal{K}$  is bounded in  $L^0(\mu)$ , it must be bounded in  $L^0(\lambda)$  as well.  $\square$

Generally speaking, the convexity property cannot be dropped as the convex hull of an  $L^0(\lambda)$  bounded set need not be bounded. Likewise, it seems not possible to relax significantly the lower boundedness assumption.

The first implication regards the possibility of replacing  $L^0$  continuity with  $L^1$  continuity under a measure change.

**Corollary 3.** *Let  $X$  be a locally convex topological vector space,  $V \subset X$  a convex neighborhood of the origin and  $T : X \rightarrow L^0(\lambda)$  a continuous linear operator such that  $T[V] = \{T(x) : x \in V\}$  is lower bounded in  $L^0(\lambda)$ . There exists then  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $T[X] \subset L^1(\mu)$  and  $T : X \rightarrow L^1(\mu)$  is continuous.*

*Proof.* Let  $\mathcal{K} = T[V]$ . Then,  $\mathcal{K}$  is convex, bounded in  $L^0(\lambda)$  and lower bounded. By Theorem 2 there exists  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $\mathcal{K}$  is bounded in  $L^1(\mu)$ . Given that each neighborhood of the origin is absorbing, this implies that  $T[X] \subset L^1(\mu)$ . Moreover,  $T : X \rightarrow L^1(\mu)$  is bounded on a neighborhood of the origin and it is thus continuous.  $\square$

A subset  $U$  of a vector lattice  $X$  is solid if  $x \in U$ ,  $y \in X$  and  $|y| \leq |x|$  imply  $y \in U$ .

**Corollary 4.** *Let  $X$  be a vector lattice with a convex, solid topological basis. A positive, continuous operator  $T : X \rightarrow L^0(\lambda)$  admits  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $T[X] \subset L^1(\mu)$  and  $T : X \rightarrow L^1(\mu)$  is continuous.*

*Proof.* Let  $V$  be a convex, solid neighborhood of the origin on which  $T$  is bounded,  $V_+ = V \cap X_+$  and let  $\mathcal{K} = T[V_+]$ . By Theorem 2 there is  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $\mathcal{K}$  is bounded in  $L^1(\mu)$ . If  $x \in V$ , then  $|T(x)| \leq T(|x|) \in \mathcal{K}$ . Thus  $T[V]$  is a bounded subset of  $L^1(\mu)$ .  $\square$

This last Corollary applies, e.g., to the space  $X = \mathfrak{B}(S)$  of bounded functions on some set  $S$  (endowed with the supremum norm).

**Corollary 5.** *Any positive linear operator  $T : \mathfrak{B}(S) \rightarrow L^0(\lambda)$  admits  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $T[\mathfrak{B}(S)] \subset L^1(\mu)$  and that  $T : \mathfrak{B}(S) \rightarrow L^1(\mu)$  is continuous. If  $\lambda \in ca(\mathcal{A})$  then  $T : \mathfrak{B}(S) \rightarrow L^0(\lambda)$  is continuous too.*

*Proof.* The unit ball  $V$  around the origin is mapped into the set  $T[V] \subset [T(-1), T(1)]$  which is bounded in  $L^0(\lambda)$  and admits  $T(-1)$  as a lower bound. By Theorem 2 there is  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $T[V]$  is bounded in  $L^1(\mu)$  so that  $T : \mathfrak{B}(S) \rightarrow L^1(\mu)$  is continuous. If  $\lambda \in ca(\mathcal{A})$ , then  $T[V]$ , being bounded in  $L^1(\mu)$  is also bounded in  $L^0(\mu)$  and thus in  $L^0(\lambda)$  as  $\mu$  and  $\lambda$  are equivalent.  $\square$

**Theorem 3.** *Let  $\Sigma$  be an algebra of subsets of some set  $S$ ,  $\mathcal{S}(\Sigma, \mathcal{A})$  the space of  $\Sigma$ -simple functions with coefficients in  $\mathcal{S}(\mathcal{A})$  and  $F : \Sigma \rightarrow L^0(\lambda)$  a vector measure. If the writing*

$$(3) \quad \int f dF = \sum_{n=1}^N f_n F(H_n) \quad f = \sum_{n=1}^N f_n \mathbf{1}_{H_n} \in \mathcal{S}(\Sigma, \mathcal{A})$$

*implicitly defines a continuous linear map of  $\mathcal{S}(\Sigma, \mathcal{A})$  into  $L^0(\lambda)$  then there exists  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that the integral  $\int f dF$  is a continuous linear mapping of  $\mathcal{S}(\Sigma, \mathcal{A})$  into  $L^1(\mu)$*

*Proof.*  $\int f dF : \mathcal{S}(\Sigma, \mathcal{A}) \rightarrow L^0(\lambda)$  is a continuous linear map if and only if the set

$$I = \left\{ \int f dF : f \in \mathcal{S}(\Sigma, \mathcal{A}), \|f\| \leq 1 \right\}$$

is bounded in  $L^0(\lambda)$ . Observe that  $J = \text{co}\{|F(H)| : H \in \Sigma\} \subset I$ . By Theorem 2 there is  $\nu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $J$  is bounded in  $L^1(\nu)$  and thus that  $I \subset L^1(\nu)$ . Moreover,  $I$  is bounded in  $L^0(\nu)$  so that, by Lemma 1, there is  $\mu \in \mathbb{P}_*(\mathcal{A}, \nu) \subset \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $I$  is a bounded subset of  $L^1(\mu)$  as claimed.  $\square$

A classical example of an operator mapping (a subspace of)  $\mathfrak{B}(S)$  into  $L^0(\lambda)$  is of course the stochastic integral  $\int h dS$  when  $S$  is a  $\lambda$  semimartingale and  $\lambda \in \mathbb{P}(\mathcal{A})$ . The preceding Corollaries thus implicitly suggest that a meaningful definition of a semimartingale may be obtained even when  $\lambda$  fails to be countably additive.

#### 4. $\lambda$ -CONVERGENT SEQUENCES.

The same measure change technique exploited above will be applied in this section to sequences. We start noting that for countable subsets of  $L^0(\lambda)$  the existence of a lower bound may be obtained by requiring a special form of  $L^0(\lambda)$  boundedness.

**Lemma 2.** *A sequence  $\langle f_n \rangle_{n \in \mathbb{N}}$  in  $L^0(\lambda)$  admits a lower bound in  $L^0(\lambda)$  if and only if*

$$(4) \quad \lim_k |\lambda|^*(\sup_{n > k} f_n^- > h) = 0 \quad \text{for some } h \in L^0(\lambda)$$

*Proof.* Assume (4) and let  $g_k = \sum_{n \leq k} 2^n (f_n^- - h)^+$  and  $g = \sum_n 2^n (f_n^- - h)^+$ . Then,  $\{|g - g_k| > c\} \subset \bigcup_{n > k} \{f_n^- > h\} \subset \{\sup_{n > k} f_n^- > h\}$  so that, by the assumption,  $\langle g_k \rangle_{k \in \mathbb{N}}$   $\lambda$ -converges to  $g \in L^0(\lambda)$ . But then,  $f_n^- \leq \eta + (f_n^- - h)^+ \leq h + g$ . The converse is obvious.  $\square$

Of course (4) is a stronger condition than just  $L^0(\lambda)$  boundedness of the sequence.

The following Theorem establishes a finitely additive version of a beautiful result of Memin [8, Lemma I.4, p. 13], widely used in the theory of stochastic integration.

**Theorem 4.** *Let  $\langle f_n \rangle_{n \in \mathbb{N}}$  be a Cauchy sequence in  $L^0(\lambda)$ , i.e. such that*

$$(5) \quad \limsup_n \sup_{p, q} |\lambda|^* (|f_{n+p} - f_{n+q}| > c) = 0 \quad c > 0$$

*There exists  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  and a subsequence  $\langle f_{n_k} \rangle_{k \in \mathbb{N}}$  which is Cauchy in  $L^1(\mu)$ . If  $\langle f_n \rangle_{n \in \mathbb{N}}$  is  $\lambda$ -convergent to  $f \in L^0(\lambda)$ , then  $\langle f_{n_k} \rangle_{k \in \mathbb{N}}$  converges to  $f$  in  $L^1(\mu)$ .*

*Proof.* Choose an increasing sequence  $\langle n_k \rangle_{k \in \mathbb{N}}$  of integers such that

$$(6) \quad \sup_{p, q} |\lambda|^* (|f_{n_k+p} - f_{n_k+q}| > 2^{-k}) \leq 2^{-k}$$

and put  $g_k = 2^k |f_{n_k} - f_{n_{k+1}}|$ . Fix a sequence  $\langle \alpha_k \rangle_{k \in \mathbb{N}}$  of positive numbers summing to 1 with finitely many non null terms. Exploiting the subadditivity of the set function  $|\lambda|^*$  we obtain the

following inequality:

$$\begin{aligned}
|\lambda|^* \left( \sum_k \alpha_k g_k > 2c \right) &\leq |\lambda|^* \left( \sum_{k < k_0} \alpha_k g_k > c \right) + |\lambda|^* \left( \sum_{k \geq k_0} \alpha_k g_k > c \right) \\
&\leq |\lambda|^* \left( \sum_{k < k_0} \alpha_k |f_{n_k} - f_{n_{k+1}}| > 2^{-k_0} c \right) + \sum_{k \geq k_0} |\lambda|^* (|f_{n_k} - f_{n_{k+1}}| > 2^{-k}) \\
&\leq |\lambda|^* \left( \sup_{k < k_0} |f_{n_k} - f_{n_{k+1}}| > 2^{-k_0} c \right) + 2^{-k_0+1}
\end{aligned}$$

If  $k_0$  and  $c$  are large enough so that  $2^{-k_0+1} < \varepsilon/2$  and  $|\lambda|^*(\sup_{k < k_0} |f_{n_k} - f_{n_{k+1}}| > 2^{-k_0} c) < \varepsilon/2$ , then,  $|\lambda|^*(\sum_k \alpha_k g_k > 2c) < \varepsilon$ . In other words, the set  $\mathcal{K} = \text{co}\{g_1, g_2, \dots\}$  is bounded in  $L^0(\lambda)$ , convex and lower bounded by 0 and, by Theorem 2, there is then  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  such that  $\sup_{h \in \mathcal{K}} \mu(h) < \infty$ . But then,  $\mu(|f_{n_k} - f_{n_{k+1}}|) \leq 2^{-k} \sup_{h \in \mathcal{K}} \mu(h)$  and  $\langle f_{n_k} \rangle_{k \in \mathbb{N}}$  is Cauchy in  $L^1(\mu)$ . If  $\langle f_n \rangle_{n \in \mathbb{N}}$   $\lambda$ -converges to some limit  $f$ , then  $\langle f_{n_k} \rangle_{k \in \mathbb{N}}$   $\mu$ -converges to  $f$  by absolute continuity. It follows that  $f \in L^1(\mu)$  and that  $\langle f_{n_k} \rangle_{k \in \mathbb{N}}$  converges to  $f$  in  $L^1(\mu)$  by [6, III.3.6].  $\square$

The above Theorem 4 allows to replace measure convergence with  $L^1$  convergence. However, proving that a sequence converges in measure and identifying its limit is often a difficult step, especially if compared with the pointwise convergence criterion considered under countable additivity. A Theorem that establishes conditions under which, by a convenient change of the underlying measure, pointwise convergence induces convergence in measure is yet not available. However, a partial step in this direction is made in the following Theorem in which we prove the finitely additive version of a subsequence principle that is often useful in applications. It is related to a well known result of Komlós [7].

**Theorem 5.** *Let  $\langle f_n \rangle_{n \in \mathbb{N}}$  be a sequence in a bounded subset  $\mathcal{K}$  of  $L^0(\lambda)_+$ . There exists  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  and a sequence  $\langle g_n \rangle_{n \in \mathbb{N}}$  with  $g_n \in \text{co}\{f_n, f_{n+1}, \dots\}$  which is Cauchy in  $L^1(\mu)$ .*

*Proof.* Assume that  $\mathcal{K}$  is bounded in  $L^1(\lambda)$  and consider the sequence  $\langle \lambda_n \rangle_{n \in \mathbb{N}}$  with  $\lambda_n = \lambda_{f_n}$ . By [4, Theorem 5] there exists  $\mu_n \in \text{co}\{\lambda_n, \lambda_{n+1}, \dots\}$  such that the sequence  $\langle \mu_n \wedge k\lambda \rangle_{n \in \mathbb{N}}$  is norm convergent for all  $k \in \mathbb{R}_+$ . Let  $h_n \in \mathcal{K}_n$  be such that  $\mu_n = \lambda_{h_n}$ . Clearly,  $\lambda_{h_n \wedge k} \leq \lambda_{h_n} \wedge k\lambda$ . If  $\langle h_{n,r} \rangle_{r \in \mathbb{N}}$  is a sequence in  $\mathcal{S}(\mathcal{A})$  converging to  $h_n$  in  $L^1(\lambda)$ , then, using norm convergence,

$$\lambda_{h_n} \wedge k\lambda = \lim_r \lambda_{h_{n,r}} \wedge k\lambda = \lim_r \lambda_{h_{n,r} \wedge k} = \lambda_{h_n \wedge k}$$

the last line following from the inequality  $|x_1 \wedge k - x_2 \wedge k| \leq |x_1 - x_2|$ . Thus the sequence  $\langle h_n \wedge k \rangle_{n \in \mathbb{N}}$  is Cauchy in  $L^1(\lambda)$  for all  $k \in \mathbb{R}_+$  and, since

$$\begin{aligned}
|\lambda|^*(|h_n \wedge k - h_m \wedge k| > c) &\geq |\lambda|^*(|h_n - h_m| > c; h_m \vee h_n \leq k) \\
&\geq |\lambda|^*(|h_n - h_m| > c) - |\lambda|^*(h_m \geq k) - |\lambda|^*(h_n \geq k) \\
&\geq |\lambda|^*(|h_n - h_m| > c) - 2k^{-1} \sup_{k \in \mathcal{K}} |\lambda|(k)
\end{aligned}$$

$\langle h_n \rangle_{n \in \mathbb{N}}$  is Cauchy in  $L^0(\lambda)$ . By Theorem 4, there exists  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda)$  and a subsequence  $\langle h_{n_k} \rangle_{k \in \mathbb{N}}$ , with  $n_k > k$  which is Cauchy in  $L^1(\mu)$ . Putting  $g_k = h_{n_k}$  proves the claim. If, contrary to what initially assumed,  $\mathcal{K}$  is not bounded in  $L^1(\lambda)$  then, by Theorem 2, the above holds upon replacing  $\lambda$  with some  $\lambda_1 \in \mathbb{P}_*(\mathcal{A}, \lambda)$ . In this case  $\mu \in \mathbb{P}_*(\mathcal{A}, \lambda_1) \subset \mathbb{P}_*(\mathcal{A}, \lambda)$ .  $\square$

The claim of Theorem 5 becomes considerably stronger under countable additivity, when completeness of  $L^p$  spaces may be invoked. The sequence  $\langle g_n \rangle_{n \in \mathbb{N}}$  would then converge in  $L^1$  and, upon passing to a subsequence if necessary, a.s. too.

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